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CAN THE AVERAGE STUDENT LEARN ANALYSIS?

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Abstract

A case-study of efforts of three students trying to learn mathematical analysis is described. Concepts of concept image, concept definition, procept *and* encapsulation *are used to support the didactical strategy adopted. This strategy emphasized the propositional calculus with explicit applications of the four rules of inference in such a way as to submit the concept image to the control of the concept definition, aiming at the encapsulation of the ε–δ discourse. A detailed example is provided. Effects of the learning efforts on the students and on the faculty are discussed.*

The research question

This paper describes a case study developed jointly by one teacher, two undergraduate students in a teacher training program and one graduate student in a Mathematics Education program. The word *analysis* refers basically to the definition of limit and the construction of the real numbers. The expression *learn analysis* refers to the encapsulation of a particular process as an object. "*Average student"* refers to the students' self evaluation; they ranked themselves in the second quarter of their classes and in the second group described by Pinto & Gray [1995, p. 2-25]. Among equally ranked peers, they detected widespread rote learning. The directive research question emerged naturally from their dissatisfaction and desires: *can the average student like us learn analysis, or is this subject reserved only for the so-called "gifted" ones?*

Methodology

The group met once a week for three hours throughout 1996. The activity was part of an honors fellowship project for one of the undergraduate students, a chance to improve learning for the other, and an opportunity to review basic mathematical concepts for the graduate student. In the first meeting, methodology with respect to subject-matter, didactical strategy, meta-cognition and evaluation was established. Negotiation proceeded throughout the year.

The subject matter was dictated by the syllabus and homework of a regular one-year mathematical analysis course that the undergraduate students were taking from another teacher. In the second semester, the group decided to concentrate on a single subject: the construction of real numbers. This subject had come up several times in the first semester. The teacher suggested to taking Cauchy sequence approach in order to boost opportunities to work with epsilons and deltas. The only available Portuguese language source that describes the construction in detail happens to contain a mistake in the proof of the fundamental theorem on the completeness of the real numbers. A task was proposed to the group: *in this chapter there is a mistake; find it, give a counter example and produce a correct proof.*

A didactical strategy was chosen: instead of looking for a smooth transition from the intuitive to the formal level, a radicalization of the cut between *concept image* and *concept definition* should be tried, by training the students in semi-formal treatment of propositional

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calculus. The four rules of inference - universal and existential particularization and generalization were to be spelled out and systematically used.

Discussions about meta-learning and meta-teaching were carried out at the end of each session. They concerned the difficulties and progress of each student and the overall evaluation of the day's work. Some sessions of the first semester and all sessions of the second semester were videotaped. Some videos were viewed and discussed by the students. The way the teacher conducted the sessions and the adequacy, aim and effects of his interventions were analyzed and adjusted as the year progressed.

As for evaluation, the group agreed that a final research report should be submitted to PME-21 and partial results should be presented in local meetings during the year. In the beginning of the second semester, the undergraduate honor student was scheduled to present a purely mathematical report to her peer group and their program advisor at the end of the year. The performance of the undergraduate students in exams of the regular analysis course were also to be observed.

The theoretical framework

It was agreed that the theoretical framework to interpret data should be the conceptualization developed by the Advanced Mathematical Thinking group of PME: *concept image, concept definition, conflict factor* [Tall & Vinner, 1981], *process, concept, procept, ambiguity process-product, encapsulation* [Gray and Tall, 1994]. Coincidentally the research subject matched that of Pinto & Gray [1995] and Pinto & Tall [1996]; namely, students' misconceptions about limits, rational and real numbers, and the use of formal definitions. The difference is that these authors seek to investigate the students' existing state of knowledge and institutional conditions, while the present research tries to produce a change in the state of knowledge and to investigate the outcomes and feasibility of such an attempt. It should be qualified as a *case study* within an *action-research* approach.

The didactical strategy: "1/n XPTO 0"

In this section, the necessity of a didactical strategy that emphasizes the discontinuity between concept image and concept definition will be justified. Next, a fairly detailed description of the particular *elementary procept* that supports this strategy will be presented. Finally, a certain ambiguity of process-concept will be described as the expression of advanced mathematical thinking in analysis and will be described in terms of *encapsulation*.

The didactical strategy of continuity. The undergraduate students had been exposed to the "intuitive definition" of limit in calculus courses, and the graduate student had also been exposed to the "formal definition" in an analysis course similar to the one that the undergraduate students were taking. In the students' own opinion, they "attempted to learn definitions by rote but in the main failed to understand the underlying concepts" [Pinto & Gray, 1995, p. 2-18]. Work with them the year before [Leal et al, 1996] had produced evidence that they shared most of the "observed errors" about limits pointed out by Davis & Vinner [1986, p. 294]. These authors formulate a major question about misconceptions: "Is there a way to teach these concepts so that misleading images will not be formed? Or are these "naive" images unavoidable and will be formed no matter how the concept is taught?" [p. 285]. We add: what to do if they are already formed? According to the authors, *influence of language* is one of the sources of misconceptions about limits [p. 298]. Words such as "limit" have undue connotations, either inside or outside mathematics. In order to avoid them, Davis & Vinner report having tried unsuccessfully, or at least without clear success, to postpone the introduction not only of the concept definition, but also of the very word "limit". "The word *limit* was not introduced until after the correct mathematical concept was seemingly well established" [Davis & Winner, 1986, p. 299]. Postponing the concept definition until a reliable concept image can be formed is the same strategy pointed out by Tall & Vinner [1981] in the SMSP:

"(...) in the SMSP (...) the concept images of limits and continuity are carefully built up over the two years of the course with fairly formal concept definitions only being given at the very end. In this way the concept image is intended to lead naturally to the concept definition" [Tall & Vinner, 1981, p. 155].

We shall call such attempts the *didactical strategy of continuity*. It consists of seeking a natural transition form the concept image to the concept definition of limit by painstakingly expanding and adjusting the concept image so that it can accommodate the concept definition.

Difficulties with the didactical strategy of continuity. Since continuity strategies are dominant in almost any textbook on calculus or analysis, we may trust that they are associated with, if not the cause of:

"(...) the almost insignificant effect that a course on analysis had in changing the quality of mathematical thinking of a group of students (...). (...) despite their extensive work with real numbers, their concept image **had not expanded to take in the concept definition**" [Pinto & Gray, 1995, p. 2-18, our emphasis].

In the first meeting, the students expressed their understanding of the formal definition of limit with the following phrase: "*For any epsilon there is an N, starting from which the sequence converges*". The teacher asked: "*Do you mean that before this N the sequence might diverge?*". As the discussion progressed, the students ran into several contradictions, but the game "someone gives you an epsilon and you have to find an N such that" appeared to them to be an arbitrary caprice of the teacher. The persistence of the above phrase indicated that the students were trying to graft the concept definition onto the concept image. They were calculating limits correctly, and propositions such as "the limit of the product of a bounded sequence by a sequence converging to zero is zero" seemed completely obvious to them. When asked to produce a formal proof, they mixed phrases from their concept images with phrases from the concept definition. They soon started referring to bounded variables outside the formulas where they had been introduced. Whenever they referred to "this epsilon" in a formula such that $\forall \varepsilon\, P(\varepsilon)$, the teacher replied: "*I see no epsilon on this black-board*", and replaced the epsilon with another symbol, attempting to show that the meaning of the proposition remained unchanged. This elicited some astonishment among the students but no positive effects. The situation is well described as a *potential conflict factor* in Tall & Vinner [1981]:

“A more serious type of potential conflict factor is one in which the concept image is at variance not with another part of the concept image but with the formal concept itself. Such factors can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors **unless the formal concept definition develops a concept image** which can then yield a cognitive conflict. Students having such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous” [Tall & Vinner, 1981, p. 154, our emphasis].

The teacher made an effort to emphasize the role of definitions in mathematics but his attempt was rebuffed. The students manifested their conception of "definition" as a "*complete description*" of an object. For them, the definition of limit was simply intended to make the idea of limit "*more precise*". Asked to choose a couple of similar notions among *definition, theorem,* and *axiom*, they did not hesitate in uniting definition with either axiom or theorem. "The everyday life thought habits take over and the respondent is unaware of the need to consult the formal definition. Needless to say that, in most cases, the reference to the concept image cell will be quite successful. This fact does not encourage people to refer to the concept definition cell" [Vinner, 1991, p. 73]. The teacher tried to emphasize the arbitrary character of definitions: "*Definitions are arbitrary*. Definitions are "man made". Defining in mathematics is **giving a name**" [Vinner, 1991, p. 66, our emphasis]. However, the comparison of definition to the ritual of baptism made the students laugh a lot.

Rupture of concept image and concept definition. It seems that looking for a continuous transition such that the concept image would be progressively adjusted and would terminate by incorporating the concept definition leads to difficulties already recognized by Vinner [1991]:

"Only non-routine problems, in which incomplete concept images might be misleading, can encourage people to refer to the concept definition. Such problems are rare and, when given to students, considered as unfair. Thus, **there is no apparent force** which can change the common thought habits which are, in principle, inappropriate for technical contexts" [Vinner, 1991, p. 73, our emphasis].

If there is "no apparent force", how to unbalance students' notions? The answer to this question may be found in a previous paper by the same author: "(...) unless the formal concept definition develops a concept image which can then yield a cognitive conflict" [Tall & Vinner, 1981, p. 154]. At this point the notion of *concept definition image* comes in: "For each individual a concept definition generates its *own* concept image (...) which might (...) be called the "concept definition image" [Tall & Vinner, 1981, p. 153]. The question now becomes: how to make the *concept definition image* strong enough so that it acquires the power of redressing the whole concept image? The answer provided in this paper is: by stressing precise rules to manipulate the concept definition until an object is formed and simultaneously submitting the concept image to the control of the concept definition. This implies attributing an independent statute to the concept definition and introducing a rupture between concept image and concept definition.

The new didactical strategy. Gray & Tall [1994] characterize advanced mathematical thinking as the possibility of ambiguous use of *process* and *product* evoked by the same symbol. As for limits, the pr*ocess* is the *tendency* towards the limit and the *product* is the value of the limit:

"The notation $\lim_{x\to a} f(x)$ represents both the **process of tending to a limit** and the **concept of the value of the limit**, as does $\lim_{n\to\infty} s_n$ (...)" [Gray & Tall, 1994, p. 120, our emphasis].

"We conjecture that the dual use of notation as process and concept enables the more able to "tame the process of mathematics into a state of subjection"; instead of having to cope consciously with the duality of concept and process, the good mathematician thinks **ambiguously** about the symbolism for **product** and **process**" [Gray & Tall, 1994, p. 121, our emphasis].

The new didactical strategy consists of redefining *process* and *product* in the situation of limits, consequently aiming at another form of ambiguity. It starts recalling that the *concept definition* is a *verbal form*: "We shall regard the *concept definition* to be a form of words used to specify that concept" [Tall & Vinner, 1981, p. 152]. The *process* is then redefined as the *sequence of inferences* necessary to deal with the form of words used to specify the concept of limit (propositional calculus). The *product* is redefined as the demonstration, that is, the *effect of truth* of the discourse supported by such inferences. This means a shift of emphasis towards language, while keeping the same basic conceptualization of Advanced Mathematical Thinking.

Precisely, according to the old ambiguity, the use of the symbol "lim1/n=0" meant either a *tendency process* or a *final value*. The new ambiguity consists in using this symbol to mean, either that for every epsilon we can find an N (the process), or that the proposition "lim1/n=0" is true; that is, it can be sustained (by an epsilontic discourse) in the forum of the mathematical community (product). Indeed, whenever a mathematician claims that something is trivial, as they like to do, s/he is not thinking about the "cognitive complexity process-concept" but is exercising *this specific form* of process-product ambiguity: s/he is ready to sustain a discourse in terms of a chain of propositions. The process of (epsilontic) discourse has been encapsulated as an object (claim). In order to be realized, such a strategy should provide the formation of an *elementary procept* leading to the construction of this specific object.

"An *elementary procept* is the amalgam of three components: a *process* that produces a mathematical *object*, and a *symbol* that represents either the process or the object" [Gray & Tall, 1994, p. 121. authors' emphasis].

Having identified the process as the $\varepsilon-\delta$ discourse framed by the propositional calculus, the object became the referent produced by the discourse. Thus the aim of the didactical strategy of rupture was to attain the limit *procept* from the side of the *concept definition*. However, one point was missing: in order to complete the construction of the elementary procept, a *symbol* was necessary. The experience was that the old symbol $\lim a_n = L$ inevitably drew the students' attention towards the concept image. For them, "lim" was the signifier attached to the idea of tendency; "lim" was *the name* of the concept image. It was necessary to adopt a *name for the concept definition*. A neutral signifier was chosen to play a temporary role: XPTO. So a definition was made, and an exercise was proposed:

" a_n XPTO L means $\forall \varepsilon \, \exists N \, \forall n \left(n > N \rightarrow \left| a_n - L \right| < \varepsilon \right)$. Show that $\frac{1}{n}$ XPTO 0 "

It is necessary to stress that XPTO is not a new symbol for the limit; it is a new symbol for the definition. It is a name for the definition; not a name for the limit. It is a temporary signifier to be used, not while the concept image is not well established, but while the concept definition is not strong enough to rule the concept image. The effects of the brute force declaration of traditional analysis courses: "from now on " $\lim_{n\to\infty} a_n = L$ means this epsilontic definition", have been negative on students. Of course, this is the desired form of the final ambiguity, but it cannot be attained by overt imposition[4].

"*This has nothing to do with getting closer*", explained the teacher. "*That N that you have found was just a sketch. The proof starts now*". He meant that the concept image had to be fully controlled and redressed in terms of the concept definition. An adaptation of Rosser [1953] made it possible to take full advantage of the propositional calculus without losing sight of the mathematical meaning of the propositions. The four inference rules were made explicit and connected to language models such as the classical syllogism. The students were required to shape every homework exercise of their analysis course into this final form. All proofs had first to be "sketched" and then "written down". Image and definition were connected but each domain had its independent validity criteria. What had to be proved was put as a question and surrounded by question-marks. This allowed the proof to proceed simultaneously, progressing from the hypothesis and regressing from the

[4] Because in Lacan's terms, it implies a reorganization of the subject's *jouissance*.

thesis, allowing a step-by-step control of what remained to be proved. Concept image was evoked precisely at the moment that a constant had to be exhibited to answer a question introduced by the existential quantifier. Once the last question had been answered, the proof was complete. There was no need to rewrite it in affirmative terms. This strategy will be exemplified below, as it was presented by the students in a poster session of a work - shop in May.

Example of the XPTO strategy

Convention: ε is a positive real variable, n e N are positive integer variables. Bars over letters introduce new variables, maintaining their respective restrictions.[i]

Hypothesis: $\begin{cases} a_n \quad \text{XPTO} \quad 0 \\ \forall n \quad |b_n| \leq K \end{cases}$

Thesis: $a_n b_n \quad \text{XPTO} \quad 0$

Proof:	**Clarification for the reader**
$? \; a_n b_n \quad \text{XPTO} \quad 0 \; ?$	From the definition we have to show that:
$? \;\; \forall \varepsilon \; \exists N \; \forall n > N \;\; |a_n b_n| < \varepsilon \; ?$	Once this question is answered, the proof is finished.
Take any $\bar{\varepsilon}$ $? \;\; \exists N \; \forall n > N \;\; |a_n b_n| < \bar{\varepsilon} \; ?$	In order to show that $\forall \varepsilon \; P(\varepsilon)$ it suffices to take an arbitrary $\bar{\varepsilon}$ and show that $P(\bar{\varepsilon})$. This rule is called *existential generalization*.
$\forall \varepsilon \; \exists N \; \forall n > N \;\; |a_n| < \varepsilon$	From the hypothesis, by definition.
$\exists N \; \forall n > N \;\; |a_n| < \dfrac{\bar{\varepsilon}}{K}$	Since $\forall \varepsilon \; P(\varepsilon)$ holds and since $\dfrac{\bar{\varepsilon}}{K} > 0$, in particular $P(\dfrac{\bar{\varepsilon}}{K})$ also holds. This rule is called *universal particularization*; it is the form of the classical syllogism: every man is mortal, Socrates is a man, hence Socrates is mortal.
Let $\bar{N}$ be such that $\forall n > \bar{N}, \quad |a_n| < \dfrac{\bar{\varepsilon}}{K}$	Since $\exists N \; P(N)$ holds, we can count on a particular $\bar{N}$ such that $P(\bar{N})$. This rule is called *existential particularization*.
$? \quad \forall n > \bar{N} \;\; |a_n b_n| < \bar{\varepsilon} \; ?$	Since, for such $\bar{N}$ we have $P(\bar{N})$, we may conclude that $\exists N \; P(N)$, answering the last question. This rule is called *existential generalization*.
Take any $\bar{n} > \bar{N}$ $? \;\; |a_{\bar{n}} b_{\bar{n}}| < \bar{\varepsilon} \; ?$	By universal generalization, it suffices to answer the question for this $\bar{n}$.
$|a_{\bar{n}} b_{\bar{n}}| = |a_{\bar{n}}| |b_{\bar{n}}| < \dfrac{\bar{\varepsilon}}{K} K = \bar{\varepsilon}$	From (9) and from the hypothesis, by universal particularization.

Results and discussion

The first question that should be asked is the following: did it work? The undergraduate students passed their analysis course, but this is not a reliable parameter; many who apparently ranked below them also passed. However the honor student made a mathematics-style exposition to another teacher in the mathematics department about the completeness of the real numbers defined in terms of equivalence classes of Cauchy sequences, which is a fairly involved $\varepsilon-\delta$ subject. "*She was self-confident on that epsilontic stuff*", he reported. On another occasion the students reported: "*Now we know in which formula to enter with $\varepsilon/3$ and where to pick the δ from. When the teacher does it, we can follow her, but when she doesn't we can't avoid filling in the gaps*." When the students were writing the final mathematical report to the honors program, they reported: "*We had trouble refraining ourselves from applying the inference rules at every instance of the resumes of previous results that did not form part of the main body of the paper. Otherwise we would never end it*." From such reports, it seems that they are playing with the $\varepsilon-\delta$ discourse as a new toy. They still cannot take it for granted and move on, but the encapsulation of the $\varepsilon-\delta$ discourse seems at its final phase. They only have to say "this is trivial", as mathematicians do.

This is the final stage of a long process. The teacher led the students to complete some formal proofs of exercises that they had done in the analysis courses. They immediately recognized the power of the method and tried to imitate it. However, at the beginning the students tried to use the inference rules prematurely, before the sketch had sufficiently been worked out. In the meetings, several times the students lost sight of the sketch at the very end of the formal proof, and the whole story had to be repeated. Some sessions lasted for more than three hours. At a certain moment, in June, the teacher requested: "*Forget about the formal proofs for the next three weeks and concentrate on the sketches*". At that moment it was not clear that the strategy would work.

Of course, it can be argued that if the same time and effort had been dedicated to the classical continuity strategy, the same result would have been attained. However the story of this case shows that such a strategy had failed before, and it would have been difficult for the students to find *affective energy* to engage in it. On the other hand, the XPTO worked not only as a symbol for the $\varepsilon-\delta$ definition but also as a brand for the group. When the students first showed the strategy in a poster session of a workshop for students and faculty, despite their efforts to the contrary, some faculty members received the XPTO as an unnecessary new symbol for the limit. A concealed similar point of view was also expressed by some of their colleagues. This made them angry. They believed in what they were doing and they wanted to show it to people. They felt they were the pioneers of the new strategy, not the underdogs of the old one. This was the *affective energy* that motivated them throughout the year.

The students evaluated the attitude of such faculty members. "*They looked irritated at the XPTO. It seems they do not want to take into account that students may have difficulties in analysis*" one of them said. Later in the year, a video of one of the sessions was shown to the teacher of the analysis course. Her first reaction was: "*But this cannot be done in a regular classroom*". The students connected this episode with the first and concluded: "*If our strategy works, they seem to feel obliged to use to it. This is a threat to their old habits*".

Actually, up to the end of October the encapsulation of the inference rules into a single object had not occurred. The existential particularization had simply been abandoned in several proofs. The connection of the rules with everyday language situations had been lost. The concept image was getting loose and recovering control over the concept definition. At this moment, the teacher calmly reminded the students: "*Next month you are going to expose this to the faculty. They will certainly ask you about the apologetic poster session of last May when you claimed that these rules were so important. What are you going to answer?*" He suggested: "*Perhaps you should tell them that our* *strategy did not work and make a traditional mathematical exposition as they like you to do*".

This remark had a decisive effect. The students started scheduling appointments among themselves in order to prepare for the exposition. The fact that they could not trust the book but, on the contrary, had to find a mistake in it, made them to become independent from the teacher. They assumed that the fight for understanding and making themselves understood was theirs. The *demand* produced by this kind of situation is well known to everyone who has learned mathematics. So it can certainly be argued that all that the XPTO strategy did was to install a certain pressure. We agree. But, was there any other way to do it?

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[i] **Theorem VI.4.2.** If P_1, P_2, ... , P_n, Q, are statements, not necessarily distinct, and x is a variable which has no free occurrences in any of P_1, P_2, ... , P_n, and if P_1, P_2, ... , P_n, Ã Q, then P_2, ... , P_n, Ã (x) Q [Rosser, 1953, p. 106].**Theorem VI.6.8.** Let x and y be variables and P and Q be statements. Let Q be the result of replacing all free occurrences of x in P by occurrences of y and P be the result of replacing all free occurrences of y in Q by occurrences of x. Then: Ã (x) $F(x) \equiv$ (y) $F(y)$ [Rosser, 1953, p. 121].