MORE GAMES FOR INTEGERS [1]

Patrícia Rosana Linardi and Roberto Ribeiro Baldino

Action-Research Group on Mathematics Education (GPA) - UNESP - Rio Claro, SP, Brazil[2]

ABSTRACT

This is the final research report of Games for integers: conceptual or semantic fields *[Souza et al. 1995]. In that paper we reported on the first applications of three games: the butterflies, the gains-and-losses, and the snails games. The last one turned out to be somewhat awkward for use in the classroom and was substituted by two others: the macaws game and the game of bets. In designing the games we have been guided by Vergnaud's theory of Conceptual Fields and sought a didactic strategy that could lead the student to provide his/her own explanation for the sign rule. The objective of this paper is to report on a systematic application of these four games in real classrooms and discuss their didactical and pedagogical value.*

1. Introduction

Negative numbers have scarcely been dealt with in recent literature on mathematics education. Among fifty-six research reports presented in PME-18, only one explicitly concerns integers [Lytle, 1994]. In PME-19, there were two out of seventy-seven [Borba, 1995; Souza et al., 1995], one out of 160 in PME-20, [Bruno & Martinon, 1996]; and none in PME-21. Negative numbers have seldom been dealt with as a topic in algebra [VIENNOT, L. 1980]. In algebraic treatments a single letter represents both, a number and a predicative sign incorporated in a single proceptual unit, the *unknown*. Only operative signs are written down. Looking for such a signed-number as the solution of an equation, presupposes a certain familiarity with this sort of object. Our efforts in developing this research are based on the belief that familiarity with negative numbers persist as a necessary step towards algebra.

In Souza et al [1995], "we thought of anticipating the solution to the sign rule problem as *theorems in action.* Our idea was that roles should be exchanged: the teacher should be the one to ask and the student the one to answer why *minus times minus makes plus*. The didactical strategy should lead the student to provide his/her own explanation to facts that s/he should consider as evident" [Souza et al. 1995, p. 233]. In order to reach this objective we designed three games to solve in action four problems **P1**. How to take the bigger from the smaller? (3 – 5 =...); **P2**. How to subtract a negative? (–(–3) =...); **P3.** What does "*minus ... times something" mean*? ((– 3) × ...?), **P4.** Why does minus times minus equal plus? (–2)(–3) =... At that occasion we were guided by two pedagogical beliefs rooted in two questions of Vergnaud [1990]. Belief 1: By engaging the student in *games* where the use of *theorems in action* leads to better playing strategies, we could make theorems become *theorems in action*. Belief 2: By introducing adequate worksheet activities based on the game, after it has been finished, we could make *theorems in action* become theorems.

• [1] Linardi, P. R. & Baldino, R. R. (1998). More games for integers. *Proceedings of PME22*, Vol. 3 p. 207-214. (30/06/1998)

[2] Partial support from CNPq and CAPES

One reported result of the two 1995 pilot experiences were confirmed by research carried out during 1996 and 1997: the will to win does not lead students to use composition of additive operators as a theorem in action. They adhere to the first strategy they find. They need assurance that a more abstract strategy exists, a demand to reach it and, in some cases, hints. Therefore the first belief is not confirmed.

The pilot experiences reported in Souza et al [1995] referred mostly to the additive structure of integers. Results obtained from experiences with the multiplication structure were meager and could not confirm our hope that the games would lead the students *to provide their own explanation of the sing rule in real classroom situations*. This paper reports how this outcome could be obtained. It introduces two new games to substitute for one of the games suggested by Souza et al [1995]. Since the study's conditions were quite adverse, we expect that such an outcome can be obtained more easily in normal classroom situations.

2. The multiplication structure of integers.

Historical difficulties with the conception of negative numbers are considered in Glaeser [1981]. Literature about integers provides models for the additive structure of integers [Thompson & Dreyfus, 1988] but is rather insufficient when considering the multiplication structure. Freudenthal proposes to insist on the necessity of permanence of distributive and commutative laws [Freudenthal 1973, p. 280] and to use the *geometric-algebraic permanence principle* [Freudenthal, 1983 p. 444]. After introducing the additive structure of integers through pairs of natural numbers, Dienes acknowledges the necessity of considering multiplication. He writes:

> "Before we can truly speak of spaces, we should invent at least one new operation: a mathematician would not call *space* what we have built up to now, and this because we have not invented any kind of multiplication; we have only invented multiplications and subtractions. In other words, if we start with $({}^{+}2,\ {}^{-}1)$, for instance, we should be able to double it to get $({}^{+}4,\ {}^{-}2)$ or triple it to get $({}^{+}6,\ {}^{-}3)$" [Dienes, 1972, p. 103, our translation].

However, if we look for the decisive point where the multiplication by a negative appears for the first time, we find:

> *Implicitly it is easy to see that* multiplying by negative numbers would reduce to double, triple, quadruple, etc. but this for each color separately, triangles and squares being interchanged [Dienes, 1972, p. 104, our translation, emphasis added].

It seems that this author does not give to the multiplication structure the same concrete development that he gives to the additive structure of integers. The question why minus times minus makes plus remains unanswered.

From the whole discussion we got the idea that no positive explanation ultimately exists and that some degree of arbitrariness is necessary. But, precisely where? We contend that arbitrary rule teaching is exactly the route to failure. This attempt leads the student to ask *why does minus times minus equal plus?* Then it is already too late; he has learned the solution without knowing the problem and is "fed up" with rules.

3. Four games for integers

Here is an attempt to solve problems P1 to P4. For a more complete description of the first two games we refer to Baldino [1996] and Souza et al [1995].

The butterflies game is an additive state-operator game intended to solve P1 (see figure 1).

The game of gains and losses is a real estate sales game with blue bills representing money and "red money" representing debts. It solves P2.

The macaws game is a multiplication state-operator game designed to solve P3 and P4. It consists of beads, cards and a board (see figure 3). The cards are stamped with an arrow and numbers from zero to three, preceded by + and – signs (see figures 4 and 5). The beads are blue (b) and red (r). The players must place the cards on the trajectories connecting the macaws, matching the card's arrow with the trajectory's arrow. The player who puts a card must also fill the macaws connected by the card with beads according to the following rule: If the card's arrow points from macaw M1 to macaw M2, the number of beads in M1 times the card's number must be equal to the number of beads in M2; the color of the beads in M1 must be equal to the color of the beads in M2 if the card has a plus sign and must be *different if the card has a minus sign*. Cards that result in fractional numbers of beads must not be played.

The game of bets is an auxiliary game for the macaws game, intended to be played before it. This game is designed to ease the difficulty with addition of multiplication operators, such as "four times, minus six times, is *minus two times*" (see figure 5). It consists of a board (see figure 2), instruction cards and fake money, blue representing gains and red representing losses, as in the game of gains and losses. The player follows the instruction of a randomly picked card. Here is a typical instruction: "Put your bet in A. Your first partner puts 10B (or 10R) in B. Your second partner puts the necessary amount to make the addition (or subtraction) exact in C. You collect what is put in C, your first partner collects what is put in B and your second partner collects what is put in A." The other cards contain similar instructions with the letters permuted, so that actually, in the long run, nobody wins.

4. The classroom study

Duration: The four games were used as a regular teaching device in three regular classes of a public school, 5^{th}, 6^{th}, and 7^{th} grades, of lower-middle class students, aged 10-13 years. The city was Rio Claro, SP, Brazil, where a State University (UNESP) maintains a graduate program in Mathematics Education. One of us was the teacher in charge of these classes. Mathematics classes met three days a week during six 45-minutes periods. We organized the forty students into groups of four, with a plenary meeting once a week. Grades for participation were given out every day. A pilot study with the butterflies game and the game of gains and losses carried out in 1996 had resulted in a teaching failure. After each game had been tried for a couple of weeks, we introduced worksheet activities. It turned out that the children had to resort to the cards in order to solve the activities. Most of them became uninterested. Our evaluation was that we had introduced the worksheets too soon. Therefore, in 1997 we decided that the games would be played for their own sake. We would let the *vertigo of the rule* take the students [Baudrillard. 1979]. We decided to introduce a new game only after the students produced some evidence that they were tired of the old one. We decided not to worry about the syllabus. Only at the end of the experience, if we had time, would we introduce worksheets and go on to mathematical notation.

The following chart came out:

Study duration (in days)			
Duration / Game	play	activities	total
Butterflies	colors: 6 signs: 7	3 8	9 15
Gains and losses	8	1	9
Game of bets	9	6	15
Macaws game	8	7	15
Total	38	25	63 days

The population: The conditions for the study were extremely adverse. In 1997, a government measure decreed that students could not be given failing grades. Unless the student drops out of school, credit was to be automatic. A final high school exam will be instituted in the near future. This measure only hides the symptoms of a deteriorating situation that has been going on for quite some time. In the schools of this region, it is usual to find children who cannot write their names being passed up to the third grade and adolescents finishing high school without being able to extract any meaning from what they read. Students are generally uninterested. Some schools are ruled by gangs. Destruction and stealing of school material is frequent. It is usual to dismiss classes due to teachers' absence. Students gladly go home earlier. Any motive is good enough to dismiss classes: holidays during the week, parties, parades, teachers' meetings, strikes... As a result, about 30 out of 60 school days were lost and we had to extend the scheduled duration of our study from one to two semesters. Traditionally, fulfillment of syllabuses are only questioned when the student gets a failing grade and the parents file a complaint against the teacher in the school district. Then the general verdict is the following: "If your have not covered the whole syllabus, you cannot assign a failing grade to the student".

As a consequence, the students that we found at the beginning of our study were completely unmotivated. They were used to copy unproductive tasks from the blackboard under an authoritarian look from the teacher. The physics teacher of a later grade told us that she gave up assigning problems to the students because they started up calculations without reading them and kept asking her: *Is this the way to do this one?* We could identify three large groups in the classroom. A few students (group 1) take the school seriously, as though in a stereotyped way. Other students (group 2) apparently take pleasure in challenging the teacher's authority, in spite of knowing that she is impotent to punish them. The majority (group 3) does not engage in either of these two strategies. It seems that the status obtained from being at school is sufficient for them. They love taking home high grades at the end of the year, in spite of knowing that they would get them anyhow. It seems that keeping the game of authority and pass/fail going is the major concern of all.

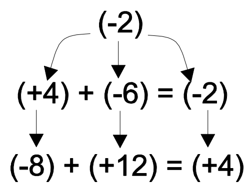
Worksheets: After the four games had been played for 38 days, the worksheet activities followed for another 25 days. Worksheets schematically reproduce parts of the game's board, such as in figures 4 and 5. For the macaws game, the task consists of reconstructing two of the following representations, given the third one: the *diagram*, the calculation *according to the cards, and* the calculation *according to the beads*. The students recognized this last modality as their natural way of playing and used it to check the others. Finally written situations that did not have a direct counterpart in the game were introduced.

5. Results

A research report should not be a bulletin of victory in teaching. Quality of research should not be confused with quality of teaching. Nonetheless, at least considering the extremely adverse teaching situation, our report is positive. We faced two challenges: a

Sample of a worksheet

The diagram:



Calculation from the cards: (-2)x((+4) + (-6)) = (-2)x(-2)=(+4)

Calculation from the beads: (-2)x((+4) + (-6)) = (-2)x(+4)+(-2)x(-6)= (-8)+(+12)=(+4)

pedagogical and a didactical one. The pedagogical challenge was to keep children in school working on some productive cultural activity. The didactical challenge was to teach negative numbers so as to arrive at the activities proposed in the textbooks. We think that both challenges were met.

We initially explained the children how we intended to work and how grades would be assigned: part from a written final and part from engagement in the games during classes. This introduction was sufficient to produce a deep change in the classroom: group 1 took a step back while group 2 took the leadership of the game organization. In a few days group 3 adhered. Along the study we noticed no deleterious attitudes. The following episodes reported in the teacher's diary illustrate how the pedagogical challenge was met.

Episode 1. There had been a party the day before and we were assigned to another room. The tables and chairs were all piled up against the walls. *Well*, I said, *I don't think that we will be able to play today, since the room is in poor condition...* Students protested: *Of course we can, teacher! Go with the two guys to get the game materials in the parking lot. We will fix the room*. When I came back they had already arranged the tables and chairs in the center of the room.

Episode 2. Children have been extremely careful not to lose nor damage the game' materials. They have been enthusiastic about the games. They asked me if they could make them to sell around their neighborhoods. Another reported to me that they were playing at home, with their relatives and friends.

Episode 3. The children introduced an extra rule for the game of bets: *the bet ought to be on the table before the card is drawn*. This is because some students can foresee their best bet from a glance at the card.

Episode 4. When the situation of figure 4 was represented in the worksheets, several students commented with amazement: "*Teacher, look: when the signals are equal it is plus, when they are different it is minus!*" I answered trying to appear very casual: "*Of course*".

We designed the game of bets because we found that the macaws game was difficult for us and for our colleagues with whom we made the first trial runs. However, the children reported that it was easier than the butterflies game. We suspect that they had already acquired the necessary degree of abstraction from the butterflies game, so that the macaws game looked easy.

The interest and class attendance persisted even after the games were finished and worksheets were introduced. Students solved the worksheet problems really fast and demanded new ones, almost exceeding our capacity to produce them. No extra explanations were necessary for activities that had no direct counterpart in the game diagram. We suspect that the students had already developed a certain ability to read, since we had insisted that they red the rules of each game before asking us to explain them. In some cases we even resorted to blackmail: *It is a pity that we can't go on playing, since you do not understand the rules...*

As for more facility of playing, no difference was detected between grades 5, 6 or 7. Children played equally well, they needed the same amount of explanations, they took approximately the same time to engage in the games and they requested help at the same points. The relevance of this outcome stems from the fact that official syllabuses recommend to start the study of negative numbers only from the 6th grade on. We had to work a lot but it paid off.

6. Discussion

The minus sign in the macaws has the property of changing colors. *Isn't this an arbitrary imposition of the sign rule ?* some people ask. Of course not. This is arbitrary but this *is not the sign rule !* Any game has arbitrary rules. But It is not when a negative multiplier acts on a colored bead that the sign rule comes up. The possibility of playing with operator qualities represented by signs and state qualities represented by colors definitely shows that this is not a commutative operation as in the sign rule. Performing "4-red times minus-three equals 12-blue", as in figure 4, amounts to the composition of a multiplication operator "three times" with a *change-colors operator.* The multiplication of negatives, as it stands in the sign rule, only becomes a commutative operation as the result of *composition of negative multiplication operators*, such as minus-two followed by minus three (see figure 4). It is the students who conclude that a plus-six card is required to close the circuit. This is the sign rule. *The sign rule involves two operators*, not an operator and a state. Besides, at this point, which of the two colors - red or blue - is going to play the negative sign is as yet undecided. It requires a lot of mathematical sophistication to imagine one of the colors as meaning "negative" and to see the phantasm of the sign rule in the change-color property of negative operators. Children do not have this sophistication.

It seems that we made the right decision in letting the children play for quite a long time before starting the activities. Surprisingly for us, they solved the activities quite fast and were engaged by them. What precisely have they learned with the games that made such a difference from the pilot experience the year before? We conjecture that the construction of integers "is an *operational synthesis* of multiplication and "change" that make operative and predicative signs merge together. Contrary to analytic processes, synthesis requires a deviation of attention from the object" [Baldino, 1998]. When the student comes to the point of asking *why does minus times minus equal plus?* it is already too late. S/he is centering attention on the question and the operational synthesis becomes difficult, or even impossible. On the other hand, it seems that the games have facilitated such a necessary decentralization of attention so that the whole set of operations was carried out in action, before they were made explicit. In particular, our expectation that the students would explain the sign rule to us was fulfilled, as illustrated by episode four.

As a final word, we should comment on the freedom we had to authorize ourselves to work with games regardless of what was prescribed in the syllabuses. Considering the school's situation, we decided that anything culturally engaging that we could do with these students would be better than any reverence to official façades. We do not regret it.

Here is our final question: How will these students do in algebra?

7. Bibliographical references

BALDINO, R.R. (1996). Las quatro operaciones con enteros a través de juegos. *UNO Revista de Didáctica de las Matemáticas.* Nº 7, pp. 37-59.

BALDINO, R. R. (1998). On the epistemology of integers. *Recherches en didactique des Mathématiques* (in press).

BAUDRILLARD, J. (1979) *De la Séduction.* Paris. Ed. Galilée.

BORBA, R. E. (1995). *Understanding and operating with integers: Difficulties and Obstacles.* Proceedings of PME-19, Recife, Vol. II, p. 226-233.

BROUSSEAU, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en Didactique des Mathématiques*, vol. 4, nº 2, pp. 165-198.

BRUNO, A., MARTINON, A. (1996) Beginning learning negative numbers. *Proceedings of PME-19*, València, Vol. II, p. 161-168.

DIENES. Z. P. (1972). *Nombres naturels entiers rationels*. Mathématique Vivante. Paris: O.C.D.L.

FREUDENTHAL, H. (1973). *Mathematics as an Educational Task*. Ridel, Dordrecht.

FREUDENTHAL, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Ridel, Dordrecht.

GLAESER, G. (1981). Épistémlogie des nombres rélatifs. *Recherches en Didactique des Mathématiques*, vol. 2, nº 3, pp. 303-346.

LYTLE, P. A. (1994). Investigation of a model based on the neutralization of opposites to teach integer addition and subtraction. *Proceedings of PME-18*, Lisboa, Vol. III, p. 192-199.

SOUZA, A. C. C. de, MOMETTI, A. L., Scavazza, H. A., BALDINO, R. R. (1995). Games for integers: Conceptual or Semantic Fields? *Proceedings of PME-19*, Recife, Vol. II, p 232-239.

THOMPSON, P. W. & DREYFUS, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education*, 19, 115-133.

VERGNAUD, G. (1990) *La Théorie des Champs Conceptuels*. Recherches en Didactiquers de Mathématiques, vol. 10° 2.3, p. 133-170.

VIENNOT, L. (1980) *Pratique de l'algèbre élémentaire chez les étudiants en physique*. Bulletin de l'Union des Physiciens, n. 662, p. 783-820.

The butterflies game

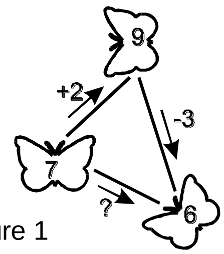


Figure 1

The game of bets

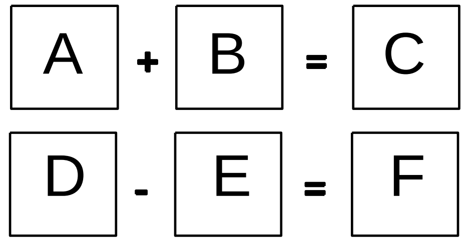
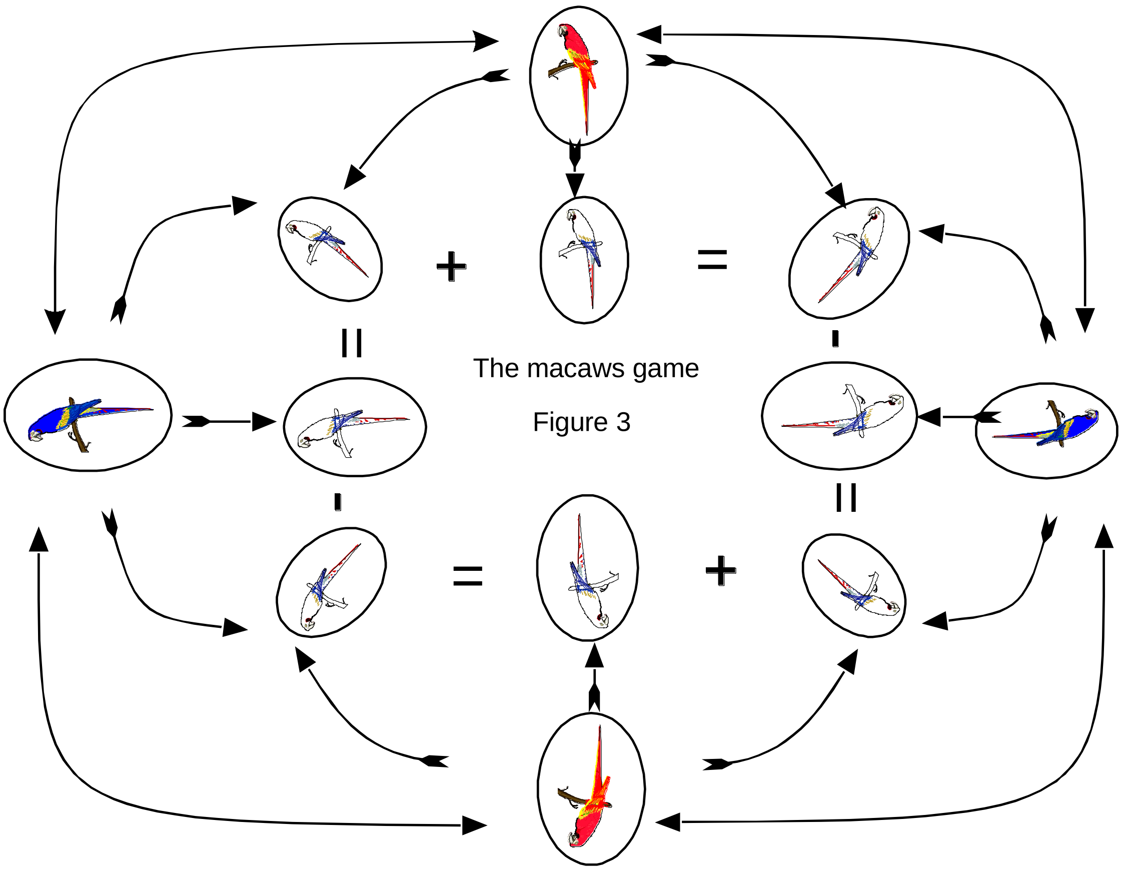


Figure 2



The macaws game

Figure 3

Creating worksheets from the macaws game

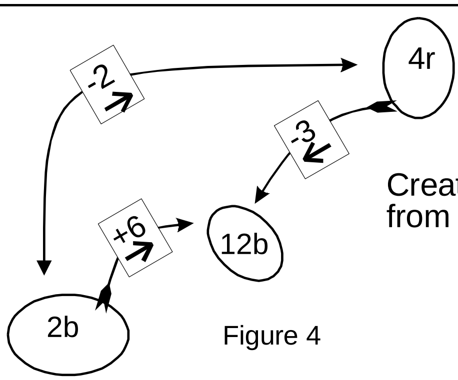


Figure 4

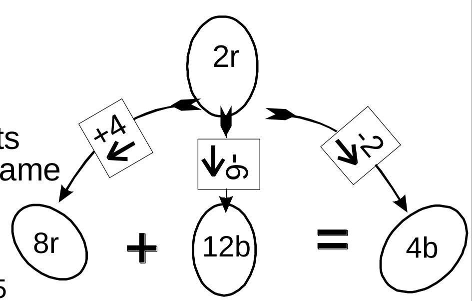


Figure 5